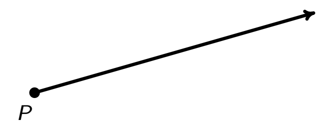
MATH 1700: SECTION B.1: ANGLES IN DEGREES

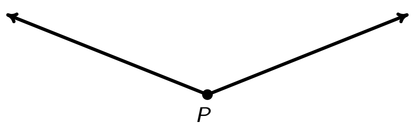
GEOMETRIC REPRESENTATION OF AN ANGLE:

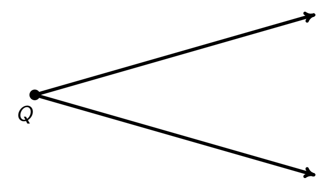
A **ray** is a 'half-line' and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other. The point from which the ray originates is called the **initial point** of the ray.



A ray with initial point P.

When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Two examples of what are commonly thought of as angles are

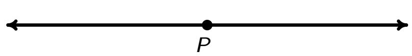


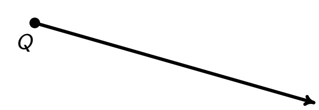


An angle with vertex P.

An angle with vertex Q.

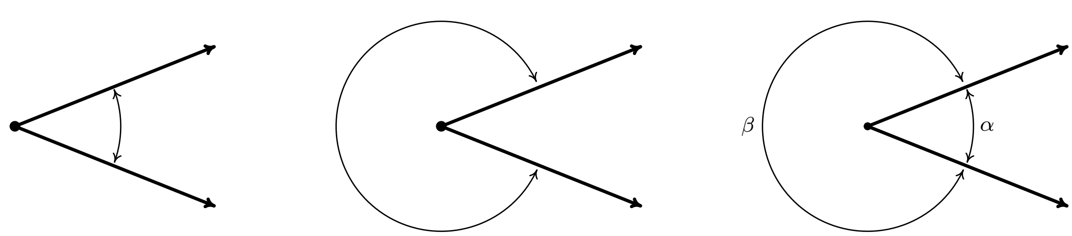
However, the two figures below also depict angles. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**; in the second, the rays are identical so the 'angle' is indistinguishable from the ray itself.





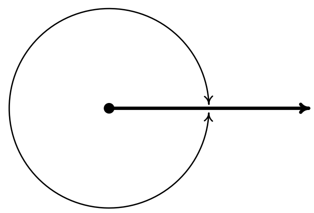
A straight angle.

The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below. Which amount of rotation are we attempting to quantify? In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles so we know more specifically which rotation we're attempting to measure.

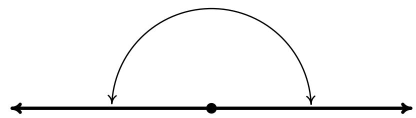


DEGREE MEASURE:

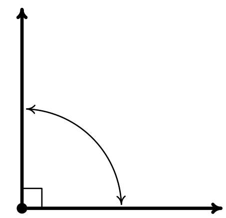
Quantities measured in **degrees** are denoted by the symbol '°.' One complete revolution is defined to be 360°, and parts of a revolution are measured proportionately. Thus half of a revolution (a straight angle) measures $\frac{1}{2}(360^\circ) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4}(360^\circ) = 90^\circ$ and so on.



One revolution $\leftrightarrow$ 360°

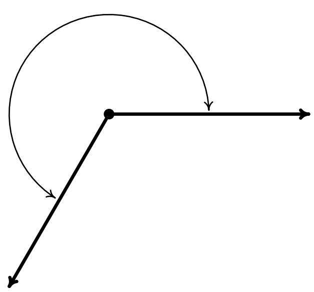


180°

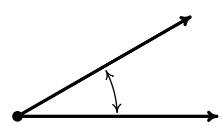


90°

If an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. It is important to note that we can know the measure of any angle as long as we know the proportion it represents of entire revolution. The measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360^\circ) = 240^\circ$, the measure of an angle which constitutes $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360^\circ) = 30^\circ$ and an angle which indicates no rotation at all is measured as 0°.



240°



30°

0°

Using our definition of degree measure, we have that 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than 1°. There are two ways to subdivide degrees. The first, and most familiar, is **decimal degrees**. For example, an angle with a measure of 30.5° would represent a rotation halfway between 30° and 31°, or equivalently, $\frac{30.5}{360} = \frac{61}{720}$ of a full rotation. This can be taken to the limit(!) using Calculus so that 'irrational' angle measures like $\sqrt{2}^\circ$ make sense.

The second way to divide degrees is the **Degree - Minute - Second** (**DMS**) system. In this system, one degree is divided equally into sixty minutes, and in turn, each minute is divided equally into sixty seconds.[1] In symbols, we write $1^\circ = 60'$ and $1' = 60''$, from which it follows that $1^\circ = 3600''$.

[1] Does this kind of system seem familiar?

EXAMPLE 1: Convert an angle with decimal measure of 42.125° to the DMS system.

We start with determining the number of whole degrees: $42.125^\circ = 42^\circ + 0.125^\circ$.

Next, we convert the partial amount of degrees to minutes: $0.125^\circ \left(\frac{60'}{1^\circ}\right) = 7.5' = 7' + 0.5'$.

Finally, we convert the partial amount of minutes to seconds: $0.5' \left(\frac{60''}{1'}\right) = 30''$. Putting it all together yields

$$\begin{aligned} 42.125^\circ &= 42^\circ + 0.125^\circ \\ &= 42^\circ + 7.5' \\ &= 42^\circ + 7' + 0.5' \\ &= 42^\circ + 7' + 30'' \\ &= 42^\circ 7' 30'' \end{aligned}$$

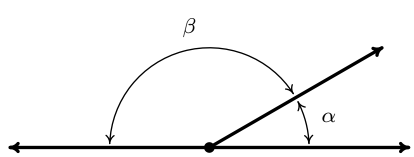
EXAMPLE 2: Convert $117^\circ 15' 45''$ to decimal degrees.

We note that $15' \left(\frac{1^\circ}{60'}\right) = \frac{15}{60}^\circ$ and $45'' \left(\frac{1^\circ}{3600''}\right) = \frac{45}{3600}^\circ$ so that:

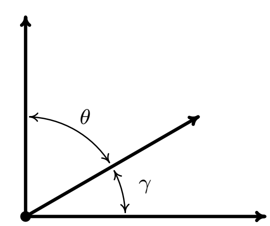
$$\begin{aligned} 117^\circ 15' 45'' &= 117^\circ + 15' + 45'' \\ &= 117^\circ + \frac{15}{60}^\circ + \frac{45}{3600}^\circ \\ &= 117^\circ + \frac{1}{4}^\circ + \frac{1}{80}^\circ \\ &= \frac{9381}{80}^\circ \\ &= 117.2625^\circ \end{aligned}$$

COMPLEMENTARY AND SUPPLEMENTARY ANGLES:

Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In the diagram below, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.



Supplementary Angles

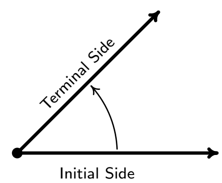


Complementary Angles

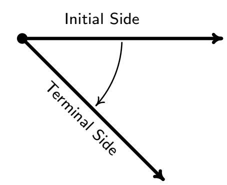
In practice, the distinction between the angle itself and its measure is blurred so that the sentence 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$.'

ORIENTED ANGLES AND STANDARD POSITION:

We now introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the **direction** of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

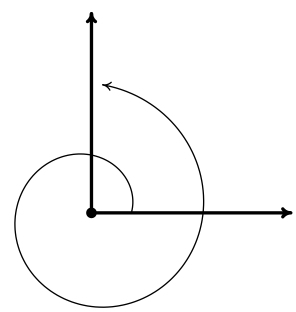


A positive angle, 45°



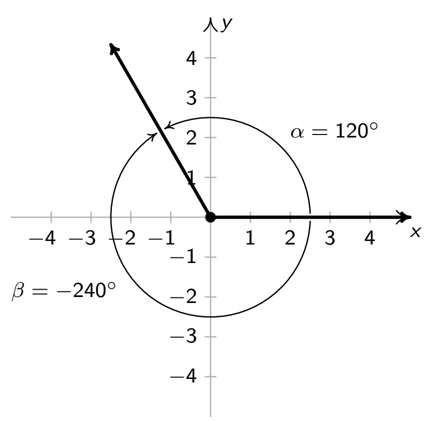
A negative angle, −45°

We also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the 'first' 360°) then continue with an additional 90° counter-clockwise rotation:



450°

An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive horizontal (usually labeled as the x-) axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a 'Quadrant I angle'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side. In the figure below, $\alpha = 120°$ and $\beta = -240°$ are two coterminal Quadrant II angles drawn in standard position.



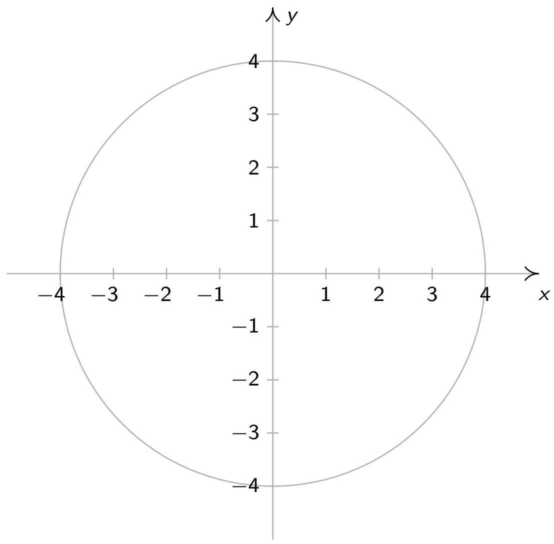
Two coterminal angles, $\alpha = 120°$ and $\beta = -240°$, in standard position.

Note that $\alpha = \beta + 360°$, or equivalently, $\beta = \alpha - 360°$.

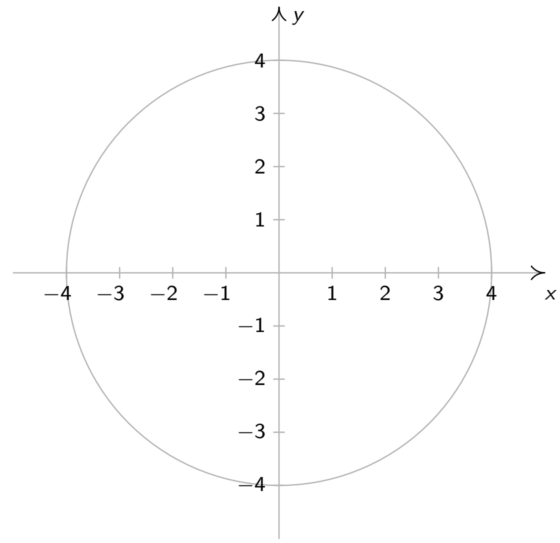
Coterminal angles always differ by a multiple of 360° (do you see why?) More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360° \cdot k$ where k is an integer (i.e., $k = 0, \pm 1, \pm 2, \ldots.$)

EXAMPLE 3: Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

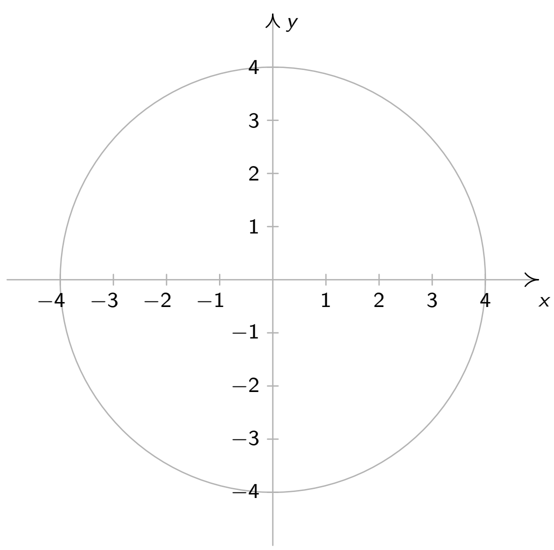
- $\alpha = 60°$



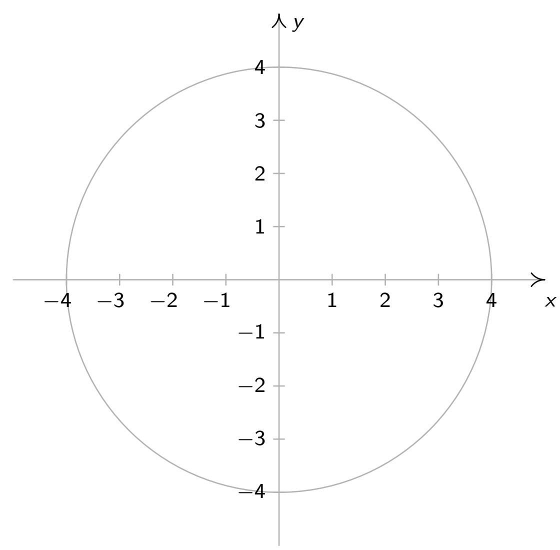
- $\beta = -225°$



- $\gamma = 540°$



- $\phi = -750°$



HOMEWORK: Appendix B.1: Page 1398: 9 - 20 all.